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Analytical calculation of massless Dirac quasi-normal modes in Schwarzschild spacetime

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Abstract

We calculate analytically the asymptotic form of quasi-normal modes of massless Dirac perturbations of a Schwarzschild black hole including first-order corrections. The spacing of the frequencies is in agreement with the case of integer spin perturbations. The real part normalized by the Hawking temperature is given by $\ln(2\cos\frac{\pi}{16})$. We also obtain explicit analytic expressions for first-order correction which is $O(n^{-1/4})$ for the nth overtone. Our results are in good agreement with existing numerical data.

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Quasi-normal modes govern the response of a black hole to external perturbations. In general, they possess a spectrum of complex frequencies due to the leakage of information into the horizon. Their observation will reveal information about the characteristics of the black hole. In asymptotically AdS spaces, they are related to properties of the dual conformal field theories on the boundary through the AdS/CFT correspondence. In asymptotically flat spaces, apart from observational possibilities, interest in QNMs has arisen [1–11] because the asymptotic form of the spectrum was shown to be related to the Barbero-Immirzi parameter [12, 13] of Loop Quantum Gravity [14–18]. The asymptotic form of the spectrum normalized by the Hawking temperature is given by

$$\frac{\omega_n}{T_H} \approx -(2n+1)\pi i + \ln 3 \tag{1}$$

for scalar and gravitational perturbations. This has been derived numerically [19–23] and subsequently confirmed analytically [3, 5]. The analytical value of the real part was first conjectured by Hod [24] based on the form of the horizon area spectrum proposed by Bekenstein and Mukhanov [25]. Its value is intriguing in Loop Quantum Gravity, suggesting that the gauge group should be $SO(3)$ instead of the expected $SU(2)$, as the latter would lead to $\Re\omega/T_H \approx \ln 2$ asymptotically.

The asymptotic expression (1) was analytically derived and generalized to arbitrary *integer* spin j [5] and a perturbative expansion was established [11]. Extending these analytical methods to half-integer spin (such as the Dirac field) is not straightforward. Dirac quasi-normal frequencies have been calculated numerically [26–35]. Here, we obtain an analytical expression of high overtones for Dirac perturbations of a Schwarzschild black hole, thus extending the results for integer spin [5, 11]. We arrive at the asymptotic expression

$$\frac{\omega_n}{T_H} \approx -\left(2n+\frac{1}{2}\right)\pi i + \ln\left(2\cos\frac{\pi}{16}\right) \tag{2}$$

Thus the spacing of the imaginary parts is the same as in the case of integer spin ($\omega_{n+1} - \omega_n \approx -2\pi i T_H$), however the real part is different. Interestingly, it is not $\ln 2$ as one might expect [25], but close (numerically, $\Re\omega/T_H \approx \ln 1.96$)!

As we shall show, the corrections are $O(n^{-1/4})$, so they are significant even for high overtones with $n \approx 100$, which makes comparison with numerical results cumbersome. Nonetheless, we obtain good agreement with numerical data [35].

We are interested in solving the massless Dirac equation in the Schwarzschild background

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \quad , \qquad f(r) = 1 - \frac{1}{r} \tag{3}$$

where we have chosen units so that the black hole mass, radius of the horizon and Hawking temperature are respectively given by

$$M = \frac{1}{2} \quad , \qquad r_0 = 1 \quad , \qquad T_H = \frac{1}{4\pi} \tag{4}$$

The wave equation may be brought into a Schrödinger-like form [28, 36],

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V[r(r_*)]\right)\Psi(r_*) = 0 \tag{5}$$

written in terms of the tortoise coordinate

$$r_* = r + \ln(1-r) \tag{6}$$

The potential is given by

$$V(r) = \mu f(r) \left(\frac{\mu}{r^2} \pm \frac{d}{dr} \sqrt{\frac{f(r)}{r^2}} \right) \tag{7}$$

where μ corresponds to a multipole number.

For quasi-normal modes, we demand the asymptotics

$$\Psi(r_*) \sim e^{\mp i\omega r_*} \quad , \qquad r_* \to \pm\infty \tag{8}$$

so that the wave is outgoing at infinity ($r_* \to \infty$) and ingoing at the horizon ($r_* \to -\infty$).

It is convenient to introduce the dimensionless coordinate $z = \omega r_*$ and expand around the black hole singularity ($z = 0$). For the potential (7) we obtain

$$\frac{1}{\omega^2} V(z) = \frac{\mathcal{A}}{z^{7/4}} + \dots \quad , \qquad \mathcal{A} = \frac{3\mu}{4(-2)^{3/4}\omega^{1/4}} \tag{9}$$

It is instructive to compare with the case of integer spin [5]. In that case,

$$\frac{1}{\omega^2} V(z) \sim \frac{1}{z^2} + \dots$$

with no leading dependence on ω. One may then take the formal limit $\omega \to \infty$ to define the zeroth-order wave equation and build a perturbative expansion in $1/\omega^a$ (where $a > 0$; it turns out that $a = 1/2$) [11]. This is not possible in the Dirac case because the coefficient of the leading-order term in the expansion of the potential (9) is ω-dependent. We end up with the wave equation

$$\frac{d^2\Psi}{dz^2} + \left[1 - \frac{\mathcal{A}}{z^{7/4}} + \dots \right] \Psi = 0 \tag{10}$$

We shall show that it is still possible to define a consistent perturbative expansion in $1/\omega^a$, if the zeroth-order equation is properly chosen (we shall show that $a = 1/4$). To this end, it will be necessary to solve the wave equation in two overlapping regimes and then match the respective solutions in the region of overlap.

First, let us solve the wave equation (10) in the *near regime* which includes the black hole singularity ($z = 0$). In this regime, we shall denote the wavefunction by a lower case ψ. We may approximate the wave equation by

$$\frac{d^2\psi}{dz^2} - \frac{\mathcal{A}}{z^{7/4}} \psi = 0 \tag{11}$$

which is valid for

$$z \lesssim \frac{1}{\omega^{1/7}} \tag{12}$$

In the near regime (12), the wave equation (11) may be solved exactly. The solution may be written in terms of Bessel functions,

$$\psi(z) = \sqrt{z} Z_4(8\mathcal{A}^{1/2} z^{1/8}) \tag{13}$$

Two linearly independent solutions are

$$\psi^{(k)}(z) = \sqrt{z} H_4^{(k)}(8\mathcal{A}^{1/2} z^{1/8}) \qquad (k = 1, 2) \tag{14}$$

At large z, they behave respectively as

$$\psi^{(1)}(z) \approx \frac{1}{2(\pi i \mathcal{A}^{1/2})^{1/2}} z^{7/16} e^{8i\mathcal{A}^{1/2} z^{1/8}} \quad , \quad \psi^{(2)}(z) \approx \frac{1}{2(-\pi i \mathcal{A}^{1/2})^{1/2}} z^{7/16} e^{-8i\mathcal{A}^{1/2} z^{1/8}} \tag{15}$$

Let us also introduce the set of independent functions

$$\begin{aligned} \psi^{(+)}(z) &= \left(\frac{\pi}{64 i \mathcal{A}^{1/2}}\right)^{1/2} \left(\psi^{(1)}(z) + i\psi^{(2)}(z)\right) \\ \psi^{(-)}(z) &= (\pi i \mathcal{A}^{1/2})^{1/2} \left(\psi^{(1)}(z) - i\psi^{(2)}(z)\right) \end{aligned} \tag{16}$$

whose large-z behavior is given respectively by

$$\psi^{(+)}(z) \approx \frac{1}{8\mathcal{A}^{1/2}} z^{7/16} \sin(8\mathcal{A}^{1/2} z^{1/8}) \quad , \quad \psi^{(-)}(z) \approx z^{7/16} \cos(8\mathcal{A}^{1/2} z^{1/8}) \tag{17}$$

These asymptotic wavefunctions in the near regime may be expanded as formal series in $\omega^{-1/4}$ (i.e., $\mathcal{A}$ (9)),

$$\begin{aligned} \psi^{(+)}(z) &= z^{9/16} - \frac{8\mu}{(-2)^{3/4}\omega^{1/4}} z^{13/16} + O(\omega^{-1/2}) \\ \psi^{(-)}(z) &= z^{7/16} - \frac{24\mu}{(-2)^{3/4}\omega^{1/4}} z^{11/16} + O(\omega^{-1/2}) \end{aligned} \tag{18}$$

Next, we turn to the *far regime* (away from the black hole singularity). It is convenient to define

$$\Psi(z) = z^{7/16} \mathcal{F}(z) \tag{19}$$

In terms of $\mathcal{F}$, the wave equation (10) reads

$$\mathcal{F}'' + \frac{7}{8z}\mathcal{F}' + \mathcal{F} = -\frac{\mathcal{A}}{z^{7/4}}\mathcal{F} \tag{20}$$

We wish to solve this equation in a region which includes $z = \infty$ and. An exact solution is not available, so we shall solve it perturbatively. The solution will be valid for arbitrarily small (as well as large) values of z (excluding the black hole singularity). Thus, the domain will overlap with the near region (12). Expanding,

$$\mathcal{F} = \mathcal{F}_0 + \omega^{-1/4}\mathcal{F}_1 + \dots \tag{21}$$

we obtain the zeroth-order wave equation

$$\mathcal{F}_0'' + \frac{7}{8z}\mathcal{F}_0' + \mathcal{F}_0 = 0 \tag{22}$$

A set of linearly independent solutions is

$$\mathcal{F}_0^{(\pm)}(z) = \mathcal{C}^{\pm} z^{\nu} J_{\pm\nu}(z) \quad , \quad \nu = \frac{1}{16} \tag{23}$$

corresponding to wavefunctions (eq. (19))

$$\Psi_0^{(\pm)}(z) = z^{7/16}\mathcal{F}_0^{(\pm)}(z) = \mathcal{C}^{\pm}\sqrt{z}J_{\pm\nu}(z) \tag{24}$$

As $z \to 0$, they behave as

$$\Psi_0^{(\pm)}(z) \approx \frac{\mathcal{C}^{\pm}}{2^{\pm\nu}\Gamma(1\pm\nu)} z^{\frac{1}{2}\pm\nu} \tag{25}$$

matching the asymptotic behavior in the near regime (18), provided we choose the constants

$$\mathcal{C}^{\pm} = 2^{\pm\nu}\Gamma(1\pm\nu) \tag{26}$$

The acceptable zeroth-order wavefunction is

$$\Psi_0(z) = \frac{i}{\pi\nu}\left(\mathcal{C}^+\Psi_0^{(-)}(z) - e^{i\nu\pi}\mathcal{C}^-\Psi_0^{(+)}(z)\right) = \sqrt{z}H_\nu^{(2)}(z) \tag{27}$$

As $z \to \infty$, it behaves as $\Psi_0 \sim e^{-iz}$, as required (eq. (8)). Indeed,

$$\Psi_0(z) \approx \mathcal{B}_0\, e^{-iz} \quad , \quad \mathcal{B}_0 = \sqrt{\frac{2}{\pi}}\, e^{i\alpha_+} \quad , \quad \alpha_\pm = \pi\left(\frac{1}{4} \pm \frac{\nu}{2}\right) \tag{28}$$

As in the bosonic case [5], we shall define the quasi-normal modes by demanding that the monodromy around the horizon be

$$\mathcal{M} = e^{4\pi\omega} \tag{29}$$

along a contour running counterclockwise encircling the singular point $r = 1$ (horizon). To compute the monodromy (29), we follow Stokes lines from $z = \infty$ to $z = -\infty$. The exact shape of the contour is immaterial, but one ought to be careful in crossing the black hole singularity $z = 0$. At the singularity $z = 0$, Stokes lines meet at an angle of 3π in the z-plane, as in the bosonic case, since the angle is solely determined by the mapping (6) (recall $z = \omega r_*$). The novel feature in the Dirac case we are discussing is that the approximate wavefunction (27) cannot be used for the requisite rotation near the singularity $z = 0$. Instead, one ought to employ the expressions in the near regime (16) which provide the exact behavior of wavefunctions near the black hole singularity. To lowest order, the acceptable wavefunction (27) in the near regime reads

$$\psi(z) = \frac{i}{\pi\nu}\left(\mathcal{C}^+\psi^{(-)}(z) - e^{i\nu\pi}\mathcal{C}^-\psi_0^{(+)}(z)\right) + O(\omega^{-1/4}) \tag{30}$$

Using the expansions (18), we see that a rotation of -3π to lowest order in $\omega^{-1/4}$ yields

$$\psi^{(\pm)}(e^{-3\pi i}z) = e^{-6i\alpha_\pm} z^{\frac{1}{2}\pm\nu} + O(\omega^{-1/4}) \tag{31}$$

where $\alpha_\pm$ are as in (28) and therefore (30) becomes

$$\psi(e^{-3\pi i}z) = \frac{i}{\pi\nu}\left(\mathcal{C}^+e^{-6i\alpha_-}\psi^{(-)}(z) - e^{i\nu\pi}\mathcal{C}^-e^{-6i\alpha_+}\psi_0^{(+)}(z)\right) + O(\omega^{-1/4}) \tag{32}$$

Next, we need to follow a Stokes line toward $z = -\infty$. To this end, we need to find expressions for the wavefunctions which are valid in a far region which includes $z = -\infty$. Working as in the far region which included $z = +\infty$ discussed above, we obtain the same form of the wave equation and the same zeroth-order wavefunctions (24). The acceptable wavefunction in this region (containing $z = -\infty$) is then deduced from (32),

$$\Psi_0(z) = \frac{i}{\pi\nu}\left(\mathcal{C}^+ e^{-6i\alpha_-}\Psi_0^{(-)}(-z) - e^{i\nu\pi}\mathcal{C}^- e^{-6i\alpha_+}\Psi_0^{(+)}(-z)\right) \tag{33}$$

Taking the limit $z \to -\infty$, we obtain

$$\Psi_0(z) \approx \sqrt{\frac{2}{\pi}}\frac{i}{\sin\nu\pi}\left(e^{-6i\alpha_-}\cos(z+\alpha_-) - e^{i\nu\pi}e^{-6i\alpha_+}\cos(z+\alpha_+)\right) \tag{34}$$

where we used (24).

Finally, to complete the contour, we rotate by π at large z which turns (34) to

$$\Psi_0(e^{i\pi}z) \approx \mathcal{D}_0 e^{-iz} + \mathcal{E}e^{iz} \quad , \qquad \mathcal{D}_0 = -2i\sqrt{\frac{2}{\pi}}\, e^{i\alpha_+}\cos\nu\pi \tag{35}$$

The other coefficient $\mathcal{E}$ may also be found but is not needed for our purposes.

By comparing the asymptotic expressions (28) and (34), we obtain the monodromy

$$\mathcal{M} = \frac{\mathcal{D}_0}{\mathcal{B}_0} + O(\omega^{-1/4}) = -2i\cos\nu\pi + O(\omega^{-1/4}) \tag{36}$$

Using (29), we deduce the asymptotic form of quasi-normal frequencies

$$\omega_n = -\left(n + \frac{1}{4}\right)\frac{i}{2} + \frac{1}{4\pi}\ln\left(2\cos\frac{\pi}{16}\right) + O(n^{-1/4}) \tag{37}$$

as promised (eq. (2) and recall $T_H = \frac{1}{4\pi}$). The real part has an interesting value. It is not $\frac{1}{4\pi}\ln 2$ as one might expect by counting fundamental degrees of freedom [25] but numerically it is close to that value. On the other hand, the spacing of the frequencies is

$$\Delta\omega_n \equiv \omega_{n+1} - \omega_n = -2\pi i T_H$$

as in the bosonic case (1).

Having obtained the zeroth-order spectrum (37), we shall calculate the first-order corrections ($O(n^{-1/4})$) next based on the method we developed in [11]. In the far regime, the first-order correction to the wavefunction (21) reads

$$\mathcal{F}_1^{(\pm)}(z) = \mathcal{F}_0^{(-)}(z)\int_0^z \frac{dx}{\mathcal{W}(x)}\mathcal{F}_0^{(+)}(x)\;\frac{-3\mu}{4(-2)^{3/4}x^{7/4}}\;\mathcal{F}_0^{(\pm)}(x) - (+\longleftrightarrow -) \tag{38}$$

in terms of the zeroth-order wavefunction (23) whose Wronskian is

$$\mathcal{W}(z) = -\frac{2\nu}{z^{1-2\nu}} \tag{39}$$

Neglecting higher-order terms in $\omega^{-1/4}$, they can be written as

$$\mathcal{F}_1^{(\pm)}(z) = \alpha^{(+\pm)}(z)\mathcal{F}_0^{(-)}(z) - \alpha^{(-\pm)}(z)\mathcal{F}_0^{(+)}(z) + O(\omega^{-1/4}) \tag{40}$$

where

$$\alpha^{(\pm\pm)}(z) = \frac{-3\mu}{8(-2)^{3/4}\nu}\int_0^z \frac{dx}{x^{3/4+2\nu}}\mathcal{F}_0^{(\pm)}(x)\mathcal{F}_0^{(\pm)}(x) \tag{41}$$

For small z, they behave as

$$\mathcal{F}_1^{(+)}(z) \approx -\frac{8\mu}{(-2)^{3/4}} z^{3/8} \quad , \quad \mathcal{F}_1^{(-)}(z) \approx -\frac{24\mu}{(-2)^{3/4}} z^{1/4} \tag{42}$$

and from (19), (21) and (25), we deduce the small-z behavior of the wavefunction which matches the expected form (18) in the near regime up to $O(\omega^{-1/2})$ corrections.

The acceptable wavefunction including first-order corrections can be written as

$$\begin{aligned} \Psi(z) &= \frac{i}{\pi\nu}\mathcal{C}^+ \left([1+\omega^{-1/4}\xi_-]\Psi_0^{(-)}(z) + \omega^{-1/4}\Psi_1^{(-)}(z) \right) \\ &\quad - \frac{ie^{i\nu\pi}}{\pi\nu}\mathcal{C}^- \left([1+\omega^{-1/4}\xi_+]\Psi_0^{(+)}(z) + \omega^{-1/4}\Psi_1^{(+)}(z) \right) \end{aligned} \tag{43}$$

correcting the zeroth-order expression (27), where (*cf.* eq. (19))

$$\Psi_1^{(\pm)}(z) = z^{7/16}\mathcal{F}_1^{(\pm)}(z) \tag{44}$$

and $\xi_\pm$ are $O(\omega^0)$ constants to be determined.

As $z \to \infty$, we have from (24)

$$\Psi_0^{(\pm)}(z) \approx \mathcal{C}^\pm \sqrt{\frac{2}{\pi}} \cos(z - \alpha_\pm) \tag{45}$$

and the first-order corrections are found from (40) and (44) to be

$$\Psi_1^{(\pm)}(z) \approx \sqrt{\frac{2}{\pi}} \left(\mathcal{C}^-\alpha^{+\pm}(\infty)\cos(z-\alpha_-) - \mathcal{C}^+\alpha^{-\pm}(\infty)\cos(z-\alpha_+) \right) \tag{46}$$

Demanding $\Psi \sim e^{-iz}$ as $z \to \infty$ imposes the constraint on the coefficients

$$\xi_- - \xi_+ + 2\alpha^{+-}(\infty) = \alpha^{--}(\infty)\frac{\mathcal{C}^+}{\mathcal{C}^-}e^{-i\nu\pi} + \alpha^{++}(\infty)\frac{\mathcal{C}^-}{\mathcal{C}^+}e^{i\nu\pi} \tag{47}$$

Then as $z \to \infty$, eq. (43) becomes

$$\Psi(z) \approx \left[\mathcal{B}_0 + \omega^{-1/4}\mathcal{B}_1 + O(\omega^{-1/2}) \right] e^{-iz} \tag{48}$$

where $\mathcal{B}_0$ is given in (28) and

$$\begin{aligned} \mathcal{B}_1 &= \sqrt{\frac{2}{\pi}}\frac{i}{\pi\nu}e^{i\alpha_+}\mathcal{C}^+ \left(\xi_-\mathcal{C}^-e^{-i\nu\pi} + \mathcal{C}^-\alpha^{+-}(\infty)e^{-i\nu\pi} - \mathcal{C}^+\alpha^{--}(\infty) \right) \\ &\quad - \sqrt{\frac{2}{\pi}}\frac{ie^{i\alpha_+}}{\pi\nu}\mathcal{C}^- \left(\xi_+\mathcal{C}^+e^{i\nu\pi} + \mathcal{C}^-\alpha^{++}(\infty) - \mathcal{C}^+\alpha^{-+}(\infty)e^{i\nu\pi} \right) \end{aligned} \tag{49}$$

Working as in the zeroth-order case, we follow the Stokes line and approach the black hole singularity ($z = 0$) where we rotate by -3π and then in reverse motion we enter a region which contains $z = -\infty$. After some algebra, we obtain the wavefunction in that region

$$\begin{aligned} \Psi(z) &= \frac{i}{\pi\nu}\mathcal{C}^+e^{-6i\alpha_-} \left([1+\omega^{-1/4}\xi_-]\Psi_0^{(-)}(-z) + \omega^{-1/4}e^{-3\pi i/4}\Psi_1^{(-)}(-z) \right) \\ &\quad - \frac{ie^{i\nu\pi}}{\pi\nu}\mathcal{C}^-e^{-6i\alpha_+} \left([1+\omega^{-1/4}\xi_+]\Psi_0^{(+)}(-z) + \omega^{-1/4}e^{-3\pi i/4}\Psi_1^{(+)}(-z) \right) \end{aligned} \tag{50}$$

correcting the zeroth-order expression (33).

Then, taking the limit $z \to -\infty$ and rotating by π to complete our excursion along the monodromy contour, we obtain

$$\Psi(e^{i\pi} z) \approx \left[\mathcal{D}_0 + \omega^{-1/4}\mathcal{D}_1 + O(\omega^{-1/2})\right] e^{-iz} + \mathcal{E} e^{iz} \tag{51}$$

where $\mathcal{D}_0$ is given in (34) and

$$\begin{aligned}
\mathcal{D}_1 &= \sqrt{\frac{2}{\pi}} \frac{i}{\pi\nu} e^{i\alpha_+} \mathcal{C}^+ e^{-6i\alpha_-} \left(\xi_- \mathcal{C}^- e^{-i\nu\pi} + e^{-3\pi i/4} \mathcal{C}^- \alpha^{+-}(\infty) e^{-i\nu\pi} - e^{-3\pi i/4} \mathcal{C}^+ \alpha^{--}(\infty)\right) \\
&- \sqrt{\frac{2}{\pi}} \frac{i e^{i\alpha_+}}{\pi\nu} \mathcal{C}^- e^{-6i\alpha_+} \left(\xi_+ \mathcal{C}^+ e^{i\nu\pi} + \mathcal{C}^- e^{-3\pi i/4} \alpha^{++}(\infty) - \mathcal{C}^+ e^{-3\pi i/4} \alpha^{-+}(\infty) e^{i\nu\pi}\right)
\end{aligned} \tag{52}$$

Again, $\mathcal{E}$ is a coefficient which may be found explicitly but is not needed for our purposes.

Comparing (48) and (51), we obtain the first-order expression for the monodromy (29)

$$\begin{aligned}
e^{4\pi\omega} &= \frac{\mathcal{D}_0 + \omega^{-1/4}\mathcal{D}_1}{\mathcal{B}_0 + \omega^{-1/4}\mathcal{B}_1} + O(\omega^{-1/2}) \\
&= \frac{\mathcal{D}_0}{\mathcal{B}_0} \left[1 + \omega^{-1/4} \left(\frac{\mathcal{D}_1}{\mathcal{D}_0} - \frac{\mathcal{B}_1}{\mathcal{B}_0}\right) + O(\omega^{-1/2})\right]
\end{aligned} \tag{53}$$

The first factor is the zeroth-order expression (36). The explicit form of the first-order correction may be found as follows. Using eqs. (28), (34), (49) and (52), after some tedious algebra we obtain

$$\begin{aligned}
\frac{4\pi\nu i \cos\nu\pi}{\mathcal{C}^+\mathcal{C}^-} \left(\frac{\mathcal{D}_1}{\mathcal{D}_0} - \frac{\mathcal{B}_1}{\mathcal{B}_0}\right) &= \left(e^{-6\pi i\nu} - e^{6\pi i\nu}\right) \alpha^{-+}(\infty) [e^{2i\nu\pi} + e^{-2i\nu\pi}] e^{-6i\nu\pi} \\
&+ \frac{\mathcal{C}^+}{\mathcal{C}^-} \alpha^{--}(\infty) [e^{3i\nu\pi} - e^{-3i\nu\pi}] e^{-6i\nu\pi} \\
&+ \frac{\mathcal{C}^-}{\mathcal{C}^+} \alpha^{++}(\infty) [e^{9i\nu\pi} - e^{-9i\nu\pi}] e^{-6i\nu\pi}
\end{aligned} \tag{54}$$

where we also used the constraint (47) to eliminate the parameters $\xi_\pm$. It turns out that the first-order correction to the monodromy is a function of the difference $\xi_+ - \xi_-$ only so the constraint (47) is sufficient to eliminate both parameters.

The coefficients $\alpha^{\pm\pm}(\infty)$ may be found explicitly by using

$$\int_0^\infty dx\, x^{-\lambda} J_\mu(x) J_\nu(x) = \frac{\Gamma(\lambda)\Gamma(\frac{\mu+\nu+1-\lambda}{2})}{2^\lambda \Gamma(\frac{\mu-\nu+1+\lambda}{2})\Gamma(\frac{-\mu+\nu+1+\lambda}{2})\Gamma(\frac{\mu+\nu+1+\lambda}{2})} \tag{55}$$

From (41) and also using (26), we obtain

$$\begin{aligned}
\frac{\mathcal{C}^-}{\mathcal{C}^+} \alpha^{++}(\infty) &= \frac{-3\mu}{8(-4)^{3/4}} \frac{\Gamma(1/16)\Gamma(3/4)\Gamma(3/16)}{\Gamma^2(7/8)} \\
\alpha^{+-}(\infty) = \alpha^{-+}(\infty) &= \frac{-3\mu}{8(-4)^{3/4}} \frac{\Gamma(1/16)\Gamma(3/4)\Gamma(1/8)}{\Gamma(13/16)\Gamma(7/8)} \\
\frac{\mathcal{C}^+}{\mathcal{C}^-} \alpha^{--}(\infty) &= \frac{-3\mu}{8(-4)^{3/4}} \frac{\Gamma(15/16)\Gamma(3/4)\Gamma^2(1/16)}{2^{3/4}\Gamma^2(7/8)\Gamma(13/16)}
\end{aligned} \tag{56}$$

To simplify (54), it helps to notice that

$$\frac{\mathcal{C}^-}{\mathcal{C}^+}\frac{\alpha^{++}(\infty)}{\alpha^{+-}(\infty)} = \frac{\Gamma(3/16)\Gamma(13/16)}{\Gamma(1/8)\Gamma(7/8)} = \frac{\sin 2\pi\nu}{\sin 3\pi\nu}$$

$$\frac{\mathcal{C}^+}{\mathcal{C}^-}\frac{\alpha^{--}(\infty)}{\alpha^{+-}(\infty)} = \frac{\Gamma(15/16)\Gamma(1/16)}{\Gamma(1/8)\Gamma(7/8)} = \frac{\sin 2\pi\nu}{\sin \pi\nu}$$

where $\nu = \frac{1}{16}$. Then (54) is easily massaged to

$$\frac{\mathcal{D}_1}{\mathcal{D}_0} - \frac{\mathcal{B}_1}{\mathcal{B}_0} = 4e^{-6i\pi\nu}\sin^2\pi\nu\ \alpha^{+-}(\infty) \tag{57}$$

Using (36) and (57), the monodromy (53) reads

$$e^{4\pi\omega} = -2i\cos\nu\pi\left[1+\sin^2\pi\nu\ \frac{3\mu}{4}\frac{\Gamma(1/16)\Gamma(3/4)\Gamma(1/8)}{\Gamma(13/16)\Gamma(7/8)}\ (4i\omega)^{-1/4} + O(\omega^{-1/2})\right] \tag{58}$$

leading to the spectrum

$$\begin{aligned}\omega_n &= -\left(n+\frac{1}{4}\right)\frac{i}{2} + \frac{1}{4\pi}\ln\left(2\cos\frac{\pi}{16}\right)\\ &\quad + \frac{3\mu}{16\pi}\sin^2\frac{\pi}{16}\ \frac{\Gamma(1/16)\Gamma(3/4)\Gamma(1/8)}{\Gamma(13/16)\Gamma(7/8)}\ (2n+1/2)^{-1/4} + O(n^{-1/2})\end{aligned} \tag{59}$$

Notice that only the real part gets corrected at $O(n^{-1/4})$. The first-order correction to the real part depends on the quantum number μ and becomes comparable to the zeroth order contribution for $n \sim 100$. Thus, for $n \lesssim 100$, higher-order corrections are significant. Nonetheless a comparison with numerical results for $n \leq 20$ [35] shows that there is pretty good agreement with our analytical expression (59) (see Fig.). The agreement is excellent on the imaginary parts for all values of μ, showing that higher-order corrections to (59) do not contribute significantly. The agreement on the real part of the frequencies becomes weaker as μ increases, because higher-order terms depend on powers of the quantum number μ.

In conclusion, we have derived analytical expressions for quasi-normal frequencies of Dirac perturbations of Schwarzschild black holes. Our calculation was based on the monodromy argument of Motl and Neitzke [5] but, unlike in the bosonic case, we had to exercise extra care as we approached the black hole singularity. We also calculated the first-order correction ($O(n^{-1/4})$) to the nth overtone based on the method we introduced in [11] and arrived at the explicit analytical expression (59). The latter is in good agreement with numerical results [35]. It would be interesting to understand the value of the real part of high overtones (2) and whether it is related in any way to the underlying number of degrees of freedom of quantum gravity [25].

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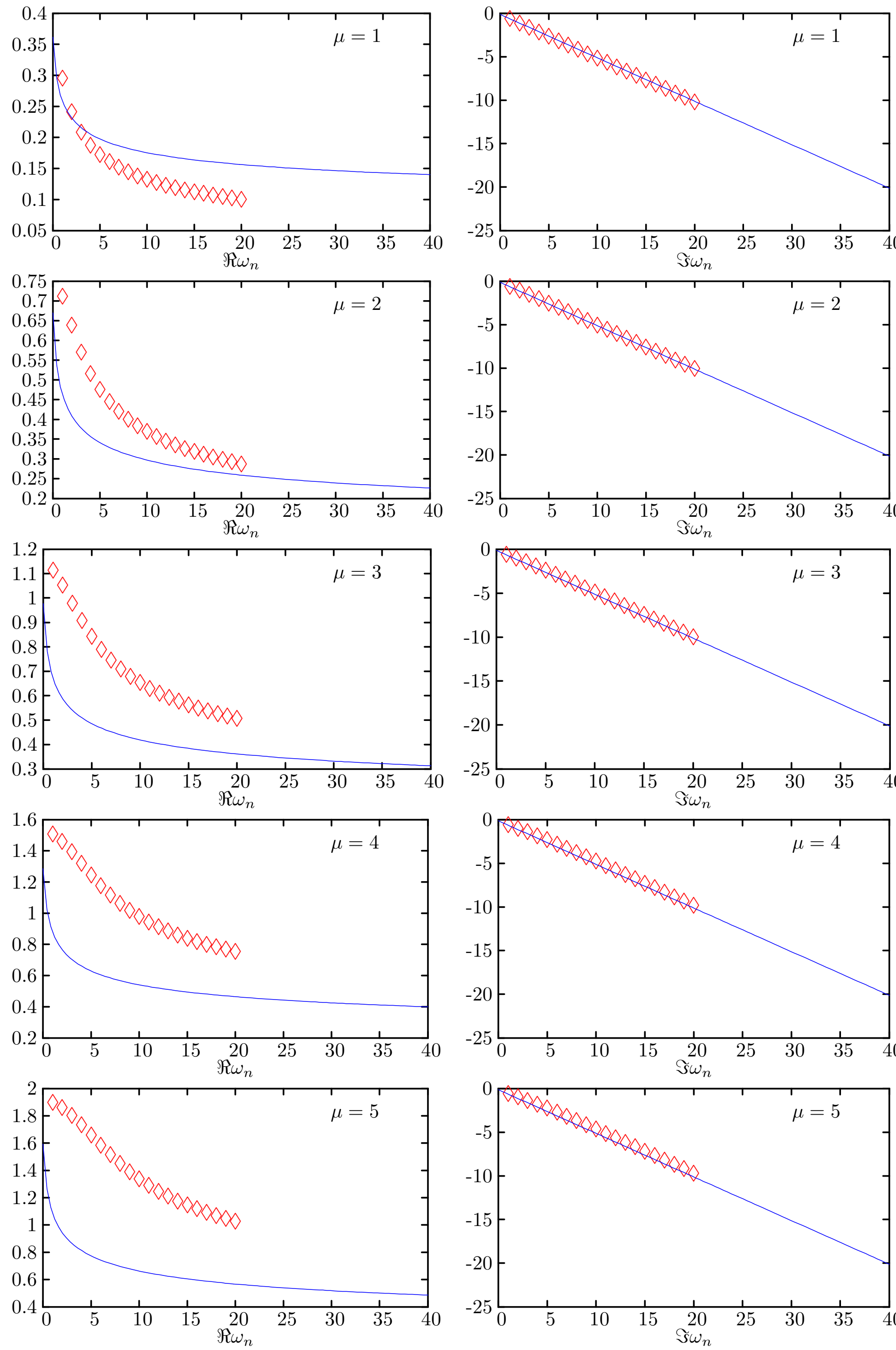

Figure 1: Quasi-normal frequencies for various values of μ: solid lines are graphs of our analytical expression (59); diamonds represent numerical data [35].